A note on the edge ideals of Ferrers graphs[1]

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Abstract We determine the arithmetical rank of every edge ideal of a Ferrers graph.

Keywords: Arithmetical rank, local cohomology, monomial ideals, edge ideals, Ferrers graphs.

Introduction

In a Noetherian commutative ring with identity, every ideal is finitely generated; in particular, for every ideal I, there are finitely many elements $f_1, \dots, f_s \in R$ such that $\mathrm{Rad}\, I = \mathrm{Rad}\,(f_1, \dots, f_s)$. The least such number s is called the *arithmetical rank* of I, denoted $\mathrm{ara}\, I$. In this paper we determine the arithmetical rank of a special class of ideals generated by squarefree monomials in a polynomial ring over a field. These are the edge ideals of certain bipartite graphs, the so-called *Ferrers graphs*, and have been considered for the first time by Corso and Nagel. This paper is intended as an application of their joint work [2], which contains an extensive treatment of the algebraic properties of the so-called *Ferrers ideals*. These admit a combinatorial description based on the *Ferrers diagrams* (also known as *Young diagrams*). We show that for every such ideal, the arithmetical rank always equals its cohomological dimension (equivalently, its projective dimension). We also give a simple characterization of the cases where the arithmetical rank coincides with the height.

On the arithmetical rank of Ferrers ideals

Let R be a commutative ring with identity. We will use the following preliminary result, which is due to Schmitt and Vogel.

Lemma 1 [[5], p.249] *Let P be a finite subset of elements of R. Let $P_1, \dots, P_r$ be subsets of P such that*

(i) $\bigcup_{i=1}^{r} P_i = P$;

(ii) P_1 *has exactly one element;*

[1]MSC 2000: 13F55; 05C99, 13D02, 13D45
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(iii) if p and p' are different elements of P_i $(1 < i \leq r)$ there is an integer i' with $1 \leq i' < i$ and an element in $P_{i'}$ which divides pp'.

We set $q_i = \sum_{p \in P_i} p^{e(p)}$, where $e(p) \geq 1$ are arbitrary integers. We will write (P) for the ideal of R generated by the elements of P. Then we get

$$\mathrm{Rad}\,(P) = \mathrm{Rad}\,(q_1, \ldots, q_r).$$

We now introduce the class of ideals that we are going to examine. Given a positive integer n, and an n-uple $\lambda = (\lambda_1, \ldots, \lambda_n)$ of positive integers such that $m = \lambda_1 \geq \cdots \geq \lambda_n$, the *Ferrers graph* $G = G_\lambda$ associated with λ is the bipartite graph on the vertex set $\{x_1, \ldots, x_n, y_1, \ldots, y_m\}$ whose set of edges is

$$E(G) = \{(x_i, y_j) | 1 \leq i \leq n, 1 \leq j \leq \lambda_i\}. \tag{1}$$

Suppose that $x_1, \ldots, x_n, y_1, \ldots, y_m$ are indeterminates over the field K. The *edge ideal* of G in the polynomial ring $R = K[x_1, \ldots, x_n, y_1, \ldots, y_m]$ is the squarefree monomial ideal

$$I(G) = \big(\{x_i y_j | (x_i, y_j) \in E(G)\}\big). \tag{2}$$

Corso and Nagel [2] call this a *Ferrers ideal*: their paper [2] is entirely devoted to the determination of the main algebraic invariants of this class of ideals. In [2], Section 2, it is shown that

$$I(G) = \bigcap_{i=1}^{n+1} (x_1, \ldots, x_{i-1}, y_1, \ldots, y_{\lambda_i}), \tag{3}$$

where, by convenience of notation, we have set $\lambda_{n+1} = 0$. Set $c_1 = 1$, and suppose that

$$\lambda_1 = \cdots = \lambda_{c_2 - 1} > \lambda_{c_2} = \cdots = \lambda_{c_3 - 1} > \lambda_{c_3} = \cdots = \lambda_{c_k - 1} > \lambda_{c_k} = \cdots = \lambda_n.$$

Finally set $c_{k+1} = n + 1$. Then a minimal prime decomposition of $I(G)$ can be obtained as follows, by omitting redundant terms from (3):

$$I(G) = \bigcap_{i=1}^{k+1} (x_1, \ldots, x_{c_i - 1}, y_1, \ldots, y_{\lambda c_i}). \tag{4}$$

The graph G can be associated with the *Ferrers diagram* having n rows and m columns, and whose rows have lengths $\lambda_1, \ldots, \lambda_n$. Note that k is the number of outer corners of the Ferrers diagram associated with G.

Proposition 1 *Let $\mu = \max_{j=1,\ldots,n}\{\lambda_j + j - 1\}$. For all $i = 1, \ldots, \mu$, set*

$$q_i = \sum_{\substack{r+s=i+1 \\ x_r y_s \in I(G)}} x_r y_s.$$

Then $I(G) = \mathrm{Rad}\,(q_1, \ldots, q_\mu)$.

Proof .-Let P be the set of minimal monomial generators of $I(G)$ given in (2), and, for all $i = 1, \ldots, \mu$, let $P_i = \{x_r y_s | r + s = i + 1, x_r y_s \in P\}$. It suffices to show that P, and $P_1, \ldots, P_\mu$ fulfill the assumption of Lemma 1. Condition (i) is true by definition of $I(G)$ (see (1) and (2)). Since $P_1 = \{x_1 y_1\}$, condition (ii) is satisfied, too. We show (iii). Let $p = x_r y_s$ and $p' = x_u y_v$ be distinct elements of P_i for some index $i = 2, \ldots, \mu$. Then $r + s = u + v = i + 1$, and, consequently, $r \neq u$, $s \neq v$. We may assume that $r < u$ and $s > v$. Then $v \leq \lambda_u \leq \lambda_r$, so that $p'' = x_r y_v \in I(G)$. More precisely, $x_r y_v \in P_{r+v-1}$, where $r + v - 1 < r + s - 1 = i$. Since p'' divides pp', this proves (iii) and completes the proof.

Proposition 1 shows that

$$\operatorname{ara} I(G) \leq \max_{j=1,\ldots,n} \{\lambda_j + j - 1\}. \tag{5}$$

As is well known (see, e.g., [3], Example 2, pp. 414–415), a lower bound for the arithmetical rank of an ideal I of R is given by its *local cohomological dimension*

$$\operatorname{cd} I = \max\{i | H^i_I(R) \neq 0\},$$

where $H^i_I(R)$ denotes the ith local cohomology module of R with respect to I. On the other hand, according to [4], Theorem 1, if I is a squarefree monomial ideal, then $\operatorname{cd} I = \operatorname{pd} I$, where pd denotes the projective dimension, i.e., the length of any minimal free resolution of I over R. Thus we have that

$$\operatorname{cd} I(G) = \operatorname{pd} I(G) \leq \operatorname{ara} I(G). \tag{6}$$

In [2] all Betti numbers of $I(G)$ are expressed by a closed formula in terms of λ. In particular we have that

$$\operatorname{pd} I(G) = \max_{j=1,\ldots,n} \{\lambda_j + j - 1\}. \tag{7}$$

From (5), (6) and (7) we derive the following

Corollary 1

$$\operatorname{ara} I(G) = \operatorname{cd} I(G) = \operatorname{pd} I(G) = \max_{j=1,\ldots,n} \{\lambda_j + j - 1\}.$$

Recall that, for every ideal I in R, $\operatorname{ht} I \leq \operatorname{ara} I$, where ht denotes the height. If $\operatorname{ara} I = \operatorname{ht} I$, I is called a *set-theoretic complete intersection*.

Corollary 2 *$I(G)$ is a set-theoretic complete intersection if and only if $\lambda = (m, m-1, \ldots, 2, 1)$.*

Proof .-It is well known that every set-theoretic complete intersection has pure height. Hence, by (4), we may assume that $m = n$. Moreover, in view of (3), $\operatorname{ht} I(G) = \min_{j=1\ldots,n+1}\{\lambda_j + j - 1\}$. Note that $m = \lambda_1 + 1 - 1 = \lambda_{n+1} + n + 1 - 1 = n$. Therefore $\operatorname{ht} I(G) = \min_{j=1\ldots,n}\{\lambda_j + j - 1\}$. On the other hand, from

Corollary 1 we have that $\operatorname{ara} I(G) = \max_{j=1\ldots,n}\{\lambda_j + j - 1\}$. Hence $I(G)$ is a set-theoretic complete intersection if and only if the minimum λ_j+j-1 coincides with the maximum λ_j+j-1, in other words, if and only if all λ_j+j-1 are equal. This occurs if and only if, for all indices $i = 1, \ldots, n$, it holds $m = \lambda_1 = \lambda_i+i-1$, i.e., $\lambda_i = m - i + 1$. This proves the claim.

The *if* part of Corollary 2 had already been proven, by different methods, in [1], Corollary 2.

Example 1 Consider the edge ideal of $G = G_\lambda$, where $\lambda = (6, 4, 4, 2, 1)$:

$$I(G) = (x_1y_1, x_1y_2, x_1y_3, x_1y_4, x_1y_5, x_1y_6, x_2y_1, x_2y_2, x_2y_3, x_2y_4,\\ x_3y_1, x_3y_2, x_3y_3, x_3y_4, x_4y_1, x_4y_2, x_5y_1).$$

We have that $\operatorname{ht} I(G) = 5$ and $\operatorname{ara} I(G) = \operatorname{cd} I(G) = 6$. Proposition 1 yields:

$$I(G) = \operatorname{Rad}(x_1y_1,\ x_1y_2 + x_2y_1,\ x_1y_3 + x_2y_2 + x_3y_1,\\ x_1y_4 + x_2y_3 + x_3y_2 + x_4y_1,\ x_1y_5 + x_2y_4 + x_3y_3 + x_4y_2 + x_5y_1,\ x_1y_6 + x_3y_4).$$

Note that the sums are taken along the ascending diagonals of the associated Ferrers diagram. In particular, the arithmetical rank can be described combinatorially as the number of such diagonals which are, at least partially, present in the diagram.

References

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